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Knot Theory Based on the Minimal Braid in Lorenz System

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Abstract By means of symbolic dynamics in Lorenz map, after studying spatial topological structure of dynamical knot constructed by the minimal braid assumption, we pry into the spatial structure of three-dimensional manifold from low-dimensional space. Lorenz dynamical knot provides a scheme about suspension. So, we are able to understand partly dynamical behaviors' topological properties of high-dimensional differential manifold by studying dynamical knot's properties. We hope to afford an approach and understand the nature of physical reality, especially in the study of DNA sequences, 20 amino acids symbolic sequences of proteid structure, and time series that can be symbolic in finance market et al.

Keywords Symbolic dynamics · Knot · Lorenz system · Minimal braid

1 Introduction

Studying the relationship of symbolic dynamics and knot theory [1] is one of the researchful directions of neoteric nonlinear science. The applications of applied symbolic dynamics [2, 3] in ordinary differential dynamical systems [4] have embodied its practicability in the researches of chaotic system. There are many close relationships between three-dimensional manifold [5, 6] defined in the ordinary differential dynamical systems and one-dimensional interval self-map, some return maps on Poincaré section of higher-dimensional

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differential manifold are usually just continuous or discontinuous alternating self-map on one-dimensional interval. Under specific parameter positions, the return map on Poincaré section of Lorenz attractor can degenerate into antisymmetric cubic map. For ordinary differential dynamical systems, it is widely used numerical experiments to simulation their spatial orbits or Poincaré section to reduce their systemic dimensions and study their dynamical behaviors. Because dynamical flow loses a degree of freedom on Poincaré section, the crux of this problem is how to find out this degree of freedom on Poincaré section by suspension outspread in three-dimensional spaces.

From symbolic dynamics, some phenomena and relationships have been discovered, which provide many new viewpoints to look at knots and symbolic sequences. It is a more natural notion that using knots to describe strange attractor, because periodic orbit is a close circle and can be regarded as a knot. It is very likely that the periodic orbit constitutes most of obvious geometric characteristics of these flows. In fact, one can use a simple method to describe the topological characteristics of knot. Starting from topological shift matrix, one can get knot by substitution braid [7], however, there still exist uncertain in this method because the knot of periodic flow constructed by the suspension outspread of the topological shift matrix is not unique. We attempt to present an assumption of the minimal braid in the substitution shift matrix, that is, the suspension outspread is natural and intuitional. This assumption has been verified by numerical experiments in Rössler attractor [8]. The numerical results in the period-doubling and knots of some low periodic sequences have proved the validity of the minimal braid assumption, and explore a new method for restoring these iterates on the section to three-dimensional information, which afford the base of numerical experiments to the method *Substitution* → *The Minimal Braid* → *Knot*. The obtainment of the minimal braid (knot) gives a more natural topological description to strange attractor.

In chaotic phenomena, for the first time, one observed strange attractor in Lorenz system, which has been the subject of study for many scientists. The knot theory has same important applications to the study of periodic orbit in Lorenz system. In the researches of chaotic phenomena, as a powerful research tool, symbolic dynamics affords a better understanding to dynamical behaviors of dynamical system. The first research of symbolic dynamics to Lorenz system was beginning from its simplified model—Lorenz map. Because Lorenz map is only a highly abstract geometric model of Lorenz system, one still cares whether it can truly reflect the essential of physics of Lorenz system. For the original Lorenz system, Lorenz map lost some information when it was simplified. It can be tested by constructing three-dimensional manifold whether the dynamical behaviors of Lorenz map accord with Lorenz system.

These researches reveal the knot theory based on the minimal braid assumption in Lorenz system. And this paper is organized as follows. In Sect. 2, introduced the form of Lorenz map. In Sect. 3, the symbolic dynamics and dual star products of Lorenz map are amplified in detail. In Sect. 4, based on the minimal knot assumption, some Lorenz knots are constructed.

2 Lorenz Map

Lorenz equation describes the convection of fluid between parallel plates, which is a typical example for chaotic systems [9]. This model has been studied extensively by numerical experiments and qualitative techniques. But its dynamics has not been fully understood yet. The corresponding equation is:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = (r - z)x - y, \\ \dot{z} = xy - bz. \end{cases}$$

In 1976, Guckenheimer and Williams for the first time introduced geometrical Lorenz model. This model is a Poincaré map on the Poincaré section which has Lorenz class manifold system. It is usually used for describing the variable regularity of atmosphere movement. The geometrical structure of Lorenz manifold may be reduced to a self-map on one-dimensional interval $f:[-\mu,\nu]\to[-\mu,\nu]$,

$$f(x)=\begin{cases} f_L(x)=\nu-\alpha|x|^\lambda+\text{h.o.t.}, & x\le 0,\\ f_R(x)=-\mu+\beta x^\lambda+\text{h.o.t.}, & x>0, \end{cases}\qquad (\mu,\nu>0,\lambda>1),$$

where λ is a constant greater than 1, "h.o.t." represents high-level term. Both of the branches f_L and f_R are monotone increasing. The generic iterative from of Lorenz map is:

$$f(x)=\begin{cases} f_L(x)=1-2|x|^2, & x\le 0,\\ f_R(x)=-1+2x^2, & x>0. \end{cases}$$

3 The Symbolic Dynamics and Dual Star Products of Lorenz Map

Seen from symbolic dynamical system, Lorenz map belongs to a more complex dynamical category because it has a discontinuity point. The Lorenz map presents more abundant dynamical actions [10] compared with Unimodal map. Let us study the symbolic dynamics of Lorenz maps [11]. Following the kneading theory [12], the address $A(x)$ of any point x on the interval $[-1,1]$ is

$$A(x)=\begin{cases} R, & x\in[-1,0),\\ L, & x\in[0,1]. \end{cases}$$

$x=0$ is the turning (discontinuous) point, and one can define C and D as

$$C=\lim_{x\to 0^-} f_L(x),$$
$$D=\lim_{x\to 0^+} f_R(x).$$

Two infinite or finite symbolic sequences starting from C and D are kneading sequences which can be ordered lexicographically by $L\prec C,D\prec R$. For two kneading sequences, $\gamma_1\gamma_2\cdots\gamma_i\gamma_{i+1}\cdots$ and $\eta_1\eta_2\cdots\eta_i\eta_{i+1}\cdots$, with maximal common leading part $\gamma_1\gamma_2\cdots\gamma_i=\eta_1\eta_2\cdots\eta_i$, one has $\gamma_1\gamma_2\cdots\gamma_i\gamma_{i+1}\cdots\prec\eta_1\eta_2\cdots\eta_i\eta_{i+1}\cdots$ if and only if $\gamma_{i+1}\prec\eta_{i+1}$. The shift operator φ is defined as $\varphi^k(\xi)=\xi_{k+1}\xi_{k+2}\cdots$ for the sequence $\xi=\xi_1\xi_2\cdots\xi_k\xi_{k+1}\cdots$. For any two sequences $\xi=\xi_1\xi_2\cdots\xi_i\xi_{i+1}\cdots$ and $\zeta=\zeta_1\zeta_2\cdots\zeta_j\zeta_{j+1}\cdots$, $\xi_i,\zeta_j\in\{R,L\}$, if $\varphi^k(\xi)\preceq\xi$ and $\zeta\preceq\varphi^k(\zeta)$, for all $K\in\mathbb{Z}_+$, then ξ is called *maximal*, ζ *minimal*, and $S=(\xi,\zeta)$ is an *extremal pair*. Let the integers k_L and k_R be the order coordinates of a letter in the sequence such that $\varphi^{k_L-1}(\xi)=L\cdots$, and $\varphi^{k_R-1}(\xi)=R\cdots$, the set k_L or k_R describe successive sequences of L or R. Then, if the pair S further satisfies the following condition:

$$\varphi^{k_L}(\xi)\preceq K^1,\qquad \varphi^{k_R}(\xi)\succeq K^2,\qquad \{k_L\}\cup\{k_R\}=\{k\}\in\mathbb{Z}_+,$$
$$\varphi^{k'_L}(\zeta)\preceq K^1,\qquad \varphi^{k'_R}(\zeta)\succeq K^2,\qquad \{k'_L\}\cup\{k'_R\}=\{k'\}\in\mathbb{Z}_+$$

S is *admissible* with respect to the kneading sequences K^1 and K^2. All the admissible pairs form an admissible set K and fill up the whole kneading parameter plane of dynamical systems of two letters.

There are two kinds of dual star products in the Lorenz map, namely, the up-star product ($\overline{*}$) and the down-star product ($\underline{*}$). Suppose there are two kneading pairs $W = (CV_1, DU_1)$ and $Z = (CV_2, DU_2)$, thereinto,

$$U_1 = u_1^1 u_2^1 \cdots u_i^1 \cdots u_k^1, \qquad V_1 = v_1^1 v_2^1 \cdots v_j^1 \cdots v_l^1, \quad u_i^1, v_j^1 \in \{L, R\},$$
$$U_2 = u_1^2 u_2^2 \cdots u_i^2 \cdots u_n^2, \qquad V_2 = v_1^2 v_2^2 \cdots v_j^2 \cdots v_m^2, \quad u_i^2, v_j^2 \in \{L, R\}$$

then, the dual star products are just $W * Z = (CV_1, DU_1) * (CV_2, DU_2)$, $* \in \{\overline{*}, \underline{*}\}$, and each symbol of Z can be replaced one by one according to the following rules,

$$\text{The up-star product}(\overline{*}): \begin{cases} C \longrightarrow CV_1RU_1, \\ D \longrightarrow DU_1LV_1, \\ L \longrightarrow LV_1RU_1, \\ R \longrightarrow RU_1LV_1. \end{cases} \tag{1}$$

$$\text{The down-star product}(\underline{*}): \begin{cases} C \longrightarrow (DU_2LV_2)^T, \\ D \longrightarrow (CV_2RU_2)^T, \\ L \longrightarrow (RU_2LV_2)^T, \\ R \longrightarrow (LV_2RU_2)^T. \end{cases} \tag{2}$$

thereinto, T is a keeping parity operator, which makes the following symbolic counterchange: $L \leftrightarrow R, C \leftrightarrow D$. Obviously, the length of compound sequences obtained by the dual star products rules (1), (2) is

$$|W| \bullet |Z| = (k + l + 2) \bullet (n + m + 2).$$

4 The Construction of Lorenz Knots

Because there are many close relationships between three-dimensional manifold and self-map on one-dimensional interval, some return maps on Poincaré section of higher-dimensional differential manifold are usually just continuous or discontinuous alternating self-maps on one-dimensional interval, for periodic orbit of Lorenz map, one can reconstruct its configuration in three-dimensional space in accordance with specific rules.

The iteration of points in periodic orbit can be expressed as braid by dynamical substitution. Here, let us present the definition of dynamical substitution, that is, let every point of period-p orbit(interval) be arranged in order, then one distributes the sequence with natural order $1, 2, 3, \ldots, p-1, p$ and replaces shifted sequence with corresponding natural order, the arrangement in order of period-p orbit(interval) points and iterative relationship are just dynamical substitution. Basing on relevant rules, one can get the braid which corresponds to its periodic orbit(symbolic sequence) [13]. In fact, the relationship between dynamical substitution and braid is one-to-many, in order to make them become one-to-one, we need add the minimal braid assumption, the minimal braid assumption can be summarized with the following two rules:

(a) The weaving rule with negative type of crossing points: if the crossings of flows arise according to the order of flow inside inner region, then the latter always goes underneath the former lines.

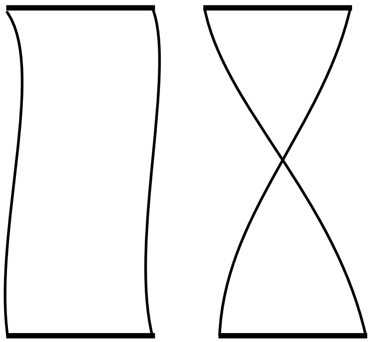

Fig. 1 Admissible cases of two flows

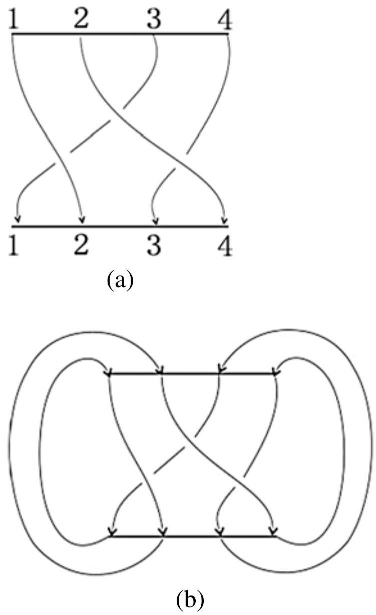

Fig. 2 Braid (**a**) and dynamical knot (**b**) correspond to symbolic sequence $RDLC$

(b) The forbidden rule for the high twist of flows: any two adjacent flows have not more than half twist (see Fig. 1).

Now, let us give some examples.

Example 1 A periodic orbit has the symbolic sequence $RDLC$, and its dynamical substitution is S_4,

$$S_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

The braid corresponds to the dynamical substitution is shown in Fig. 2a.

If one defines the region between two transversals of the braid as α, the region outside the two transversals as β, and connects these points on the transversals with curve $x \longrightarrow x$ in the region of β, another rule is that arbitrary two curves in the region of β are disjoint, one can get the projection on two-dimensional plane, which is just corresponding uniquely

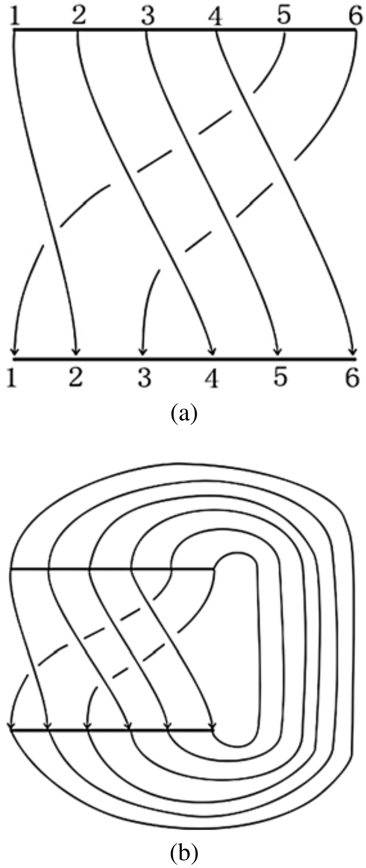

Fig. 3 Braid (**a**) and dynamical knot (**b**) correspond to symbolic sequence *RLDLLC*

to symbolic sequence. The projection of dynamical knot corresponds to symbolic sequence *RDLC* is shown in Fig. 2b.

Example 2 A periodic orbit has the symbolic sequence *RLDLLC*, and its dynamical substitution is S_6,

$$S_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 6 & 1 & 3 \end{pmatrix}.$$

The braid corresponds to the dynamical substitution is shown in Fig. 3a and the projection of dynamical knot corresponds to symbolic sequence *RLDLLC* is shown in Fig. 3b.

Example 3 A periodic orbit has the symbolic sequence *RRLDLLRC*, and its dynamical substitution is S_8,

$$S_8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 8 & 1 & 2 & 4 & 6 \end{pmatrix}.$$

The braid corresponds to the dynamical substitution is shown in Fig. 4a and the projection of dynamical knot corresponds to symbolic sequence *RRLDLLRC* is shown in Fig. 4b.

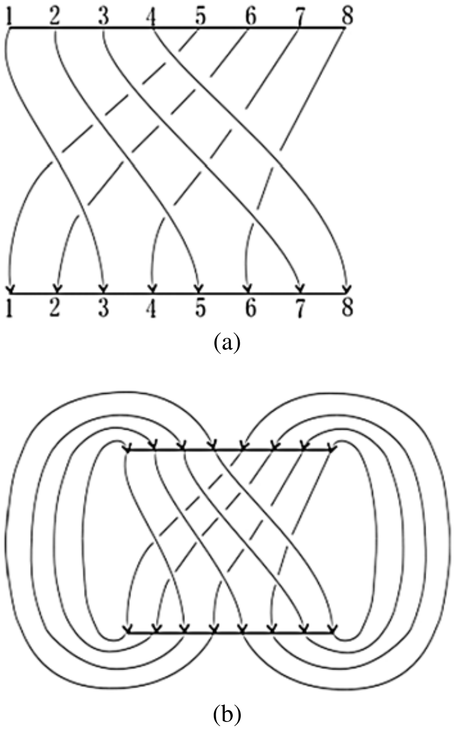

Fig. 4 Braid (**a**) and dynamical knot (**b**) correspond to symbolic sequence *RRLDLLRC*

5 Conclusion and Discussion

By means of symbolic dynamics in Lorenz map, after studying spatial topological structure of dynamical knot constructed by the minimal knot assumption, one can pry into the spatial structure of three-dimensional manifold from low-dimensional space. So, we are able to understand partly dynamical behaviors' topological properties [14] of high-dimensional differential manifold by studying dynamical knot's properties. The Lorenz dynamical knot based on the minimal knot assumption provides a scheme about suspension outspread [15]. Therefore, we put forward the minimal knot assumption, and construct dynamical knot based on the symbolic dynamics of Lorenz map. It will be a convincing proof if the knot exists in real Lorenz system, which can test whether Lorenz map is accurately reflecting the essential of physics of Lorenz system. As for the existence of Lorenz knot in real space, it is part of our future work.

We hope to afford an approach—*Substitution* → *The Minimal Braid* → *Knot* and understand the nature of physical reality, especially in the study of DNA [16–18] sequences, 20 amino acids [19, 20] symbolic sequences of proteid structure, and time series that can be symbolic in finance market et al. The establishment of this system opens up a vast vista.

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